Kuramoto Transition in Ensemble with Delayed Feedback

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A transition to collective synchrony in an ensemble of globally coupled oscillators is known as the Kuramoto transition. An important application of the theory is collective dynamics of neuronal populations. Indeed, synchronization of individual neurons is believed to play the crucial role in the emergence of pathological rhythmic brain activity in Parkinson's disease, essential tremor, and epilepsies; a detailed discussion of this topic and numerous citations can be found in Refs. [2]. One approach to suppress such an activity is to apply to the system a negative feedback loop [3, 4].

The weakly nonlinear theory of the Kuramoto transition in the presence of linear and nonlinear time-delayed coupling terms is developed. We heavily rely in our analysis on the corresponding treatment of the system without delay by Crawford [1].

1. Limit Cycle Systems ⇒ Phase Models

Our basic model is an ensemble of autonomous oscillators subject to different types of *global coupling*. We take individual oscillators as Van der Pol ones and write the model as

$$\ddot{x}_i-\mu(1-x_i^2)\dot{x}_i+\omega_i^2x_i=2\sqrt{2}\omega_i\xi_i(t)+\varepsilon' F(\bar{x},\bar{y}), \qquad (1)$$

where $\xi_i(t)$ is a δ-correlated Gaussian noise: $\langle\xi_i(t)\xi_j(t-t')\rangle=2D\delta_{ij}\delta(t')$. The ensemble averages are defined as

$$\bar{x}=\frac{1}{N}\sum_{j=1}^{N}x_j\,, \qquad \bar{y}=\frac{1}{N}\sum_{j=1}^{N}\frac{\dot{x}_j}{\omega_j}\,.$$

In the reduction to *phase equations* we use the smallness of parameters μ and ε', and suppose the natural frequencies ω_i to be distributed in a relatively close vicinity of the mean frequency $\omega_0\equiv N^{-1}\sum_{j=1}^{N}\omega_j$. Because $\mu\ll\omega_i$, the solution of the autonomous Van der Pol oscillator can be written as $x_i(t)\approx A_i(t)\cos(\varphi_i(t))$ where on the limit cycle $A_i\approx 2$ and $\dot{\varphi}_i=\omega_i$. Because $\varepsilon'\ll\mu$, coupling does not affect the amplitude, but only the phase.

The absolute value of the *complex order parameter*

$$R(t)=|R|e^{i\theta(t)}=\frac{1}{2}(\bar{x}+i\bar{y})=\frac{1}{N}\sum_{j=1}^{N}e^{i\varphi_j(t)} \qquad (2)$$

is close to 0 for nearly uniform, nonsynchronized distributions, and reaches 1 for strongly synchronized states.

Below we will be interested in linear coupling with and without time delay [3], and in a nonlinear coupling [4]:

$$\varepsilon' F(\bar{x},\bar{y}) = 2\omega_0\varepsilon\bar{y}(t)+2\omega_0\varepsilon_f\bar{y}(t-T) + \frac{d}{dt}(\bar{x}^2(t-T))(K_x\bar{x}(t)+K_y\bar{y}(t))\,.$$

As a result, the phase equations for the oscillators read

$$\dot{\varphi}_i=\omega_i+\frac{\varepsilon}{N}\sum_{j=1}^{N}\sin(\varphi_j(t)-\varphi_i(t))+\frac{\varepsilon_f}{N}\sum_{j=1}^{N}\sin(\varphi_j(t-T)-\varphi_i(t))$$
$$+\varepsilon_{of}|R|^2(t-T)|R|(t)\sin[2\theta(t-T)-\theta(t)-\varphi_i(t)+\nu]+\xi_i(t), \qquad (3)$$

where $\varepsilon_{of}e^{i\nu}=2(K_x+iK_y)$. Coupling types: (i) ε describes collective *linear coupling without delay*, as in the original Kuramoto model; (ii) ε_f describes *linear coupling with delay*, as has been proposed in [3]; (iii) ε_{of} describes *nonlinear coupling with delay*, as has been proposed in [4].

2. Linear Feedback

Thermodynamic limit and stability

In the *thermodynamic limit* $N\to\infty$ we can introduce a distribution of natural frequencies $g(\omega)$ and rewrite system (3) (here $\varepsilon_{of}=0$) as

$$\dot{\varphi}(\omega)=\omega+\varepsilon\int_{-\infty}^{+\infty}g(\omega')\sin\left(\varphi(\omega',t)-\varphi(\omega,t)\right)d\omega'$$
$$+\varepsilon_f\int_{-\infty}^{+\infty}g(\omega')\sin\left(\varphi(\omega',t-T)-\varphi(\omega,t)\right)d\omega'+\xi(\omega,t)\,. \qquad (4)$$

The distribution density $\rho(\omega,\varphi,t)$ ($\int_0^{2\pi}\rho(\omega,\varphi,t)\,d\varphi=1$) is governed by the Fokker-Planck equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\varphi}(\rho v)-D\frac{\partial^2\rho}{\partial\varphi^2}=0, \qquad (5)$$

$$v(\omega)=\omega+\varepsilon\int_0^{2\pi}d\theta\int_{-\infty}^{+\infty}d\omega' g(\omega')\sin(\theta-\varphi)\rho(\omega',\theta,t)$$
$$+\varepsilon_f\int_0^{2\pi}d\theta\int_{-\infty}^{+\infty}d\omega' g(\omega')\sin(\theta-\varphi)\rho(\omega',\theta,t-T)\,. \qquad (6)$$

The order parameter (2) takes the form

$$R(t)=\frac{1}{N}\sum_{j=1}^{N}e^{i\varphi_j(t)}=\int_{-\infty}^{+\infty}d\omega\, g(\omega)\int_0^{2\pi}d\varphi\,\rho(\omega,\varphi,t)e^{i\varphi}. \qquad (7)$$

A linear stability analysis of the absolutely nonsynchronous state $\rho_0=(2\pi)^{-1}$ reveals the only perturbations can be excited are

$$\rho_1=\frac{\varepsilon+\varepsilon_f e^{-\lambda_\pm T}}{2(\lambda_\pm+D\pm i\omega)}C_\pm e^{\pm i\varphi+\lambda_\pm t}\,; \qquad (8)$$

and the spectrum of λ_+ is formed by the roots of the *"spectral function"*

$$\Lambda(\lambda)\equiv 1-\frac{\varepsilon+\varepsilon_f e^{-\lambda T}}{2}\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{D+\lambda+i\omega}=0 \qquad (9)$$

(for λ_- the spectral function is $\Lambda^*(\lambda)$).

Generally, $\Im\left(\int_{-\infty}^{+\infty}g(\omega)(D+i\omega)^{-1}d\omega\right)=\int_{-\infty}^{+\infty}\omega g(\omega)(D^2+\omega^2)^{-1}d\omega\neq 0$; therefore real roots of $\Lambda(\lambda)$ (including $\lambda=0$) are *not* admitted and *only one complex root* $\lambda=-i\Omega$ with the corresponding mode $\rho_1=\alpha(\omega)e^{i(\varphi-\Omega t)}+cc$ determines linear stability. From the linear analysis we thus expect a *Hopf bifurcation* for the transition to synchrony.

In the degenerated case $\int_{-\infty}^{+\infty}\omega g(\omega)(D^2+\omega^2)^{-1}d\omega=0$, a relation $\Lambda^*(\lambda)=\Lambda(\lambda^*)$ holds, then real roots are admitted and complex roots appear in pairs (λ, λ^*). We expect that in real applications the degeneracy of the frequency distribution is absent, so we do not consider this situation below.

Weakly nonlinear analysis

Conventional multiple scale analysis yields

$$\rho(\omega,\varphi,t)=\frac{1}{2\pi}\left[1+\frac{\pi(\varepsilon_0+\varepsilon_f e^{i\Omega T})A_1(t)e^{i(\varphi-\Omega t)}}{D+i(\omega-\Omega)}+cc\right.$$
$$\left.+\frac{\pi^2(\varepsilon_0+\varepsilon_f e^{i\Omega T})^2A_1^2(t)e^{i2(\varphi-\Omega t)}}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}+cc+O(A_1^3)\right],$$

and for the order parameter $R(t)=2\pi A_1^*e^{i\Omega t}+O(A_1^3)$, where the amplitude A_1 obeys

$$\dot{A}_1=\lambda_2(\varepsilon_0,\Omega)A_1-P(\varepsilon_0,\Omega)A_1|A_1|^2\,, \qquad (10)$$

λ_2 is the linear growth rate

$$\lambda_2(\varepsilon,\Omega)=\frac{\varepsilon-\varepsilon_0}{i\pi\left(\varepsilon+\varepsilon_f e^{i\Omega T}\right)^2G'(\Omega+iD)+\varepsilon_f Te^{i\Omega T}}, \qquad (11)$$

$$P(\varepsilon,\Omega)=\frac{\pi^2\left|\varepsilon+\varepsilon_f e^{i\Omega T}\right|^2\left(1-\frac{G(\Omega+i2D)-G(\Omega+iD)}{iDG'(\Omega+iD)}\right)}{D\left(1+\frac{\varepsilon_f e^{i\Omega T}T}{i\pi G'(\Omega+iD)\left(\varepsilon+\varepsilon_f e^{i\Omega T}\right)^2}\right)}, \qquad (12)$$

where $G(z)\equiv\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{\omega-z}$.

Eq. (10) and the expressions (11), (12) are the main result of our analysis. They give a full description of the effect of the delayed global feedback on the synchronization transition in the ensemble of oscillators. The linear part (11) has already been discussed in [3], and the expression (12) completes the description of the synchronization transition.

For the **Lorentzian distribution** $g(\omega)=\frac{\gamma}{\pi[(\omega-\omega_0)^2+\gamma^2]}$ (γ is a characteristic width of the distribution, ω_0 is the mean frequency), $G(z)=\frac{i}{2\pi}\frac{1}{\omega_0-i\gamma-z}$, and the characteristic equation $\Lambda(\lambda=\beta+i\Omega)=0$ takes the form

$$2\Omega=2\omega_0-\varepsilon_f e^{-\beta T}\sin\Omega T, \quad \varepsilon=2(\gamma+D+\beta)-\varepsilon_f e^{-\beta T}\cos\Omega T.$$

The threshold value ε_0 is determined by $\beta=0$. Substituting the expressions above in (11),(12) we obtain

$$\lambda_2(\varepsilon_0,\Omega)=\varepsilon_2(2+\varepsilon_f Te^{i\Omega T})^{-1},$$

$$P(\varepsilon_0,\Omega)=4(\varepsilon_0+\varepsilon_f e^{i\Omega T}+2D)^{-1}(2+\varepsilon_f Te^{i\Omega T})^{-1}\,.$$

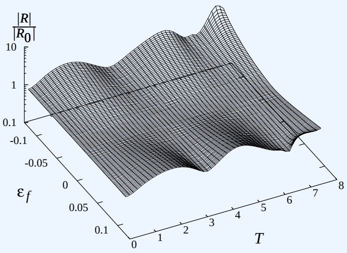

Fig. 1: Effect of delayed feedback on the order parameter for $\omega_0=1$, $\gamma=D=0.01$.

The stationary amplitude A_1 is calculated according to (10) $|A_1|^2=\Re\lambda_2/\Re P$. To demonstrate, how the delayed feedback affects the amplitude, we present in Fig. 1 the ratio $|R|/|R_0|$ where R_0 is the order parameter in the absence of delayed feedback for the same closeness to the transition point ε_2.

3. Nonlinear Delayed Feedback

For *purely* nonlinear delayed feedback the linear problem is the same as in the previous case where one sets $\varepsilon_f=0$. Therefore as soon as $g(\omega_0+\Delta\omega)=g(\omega_0-\Delta\omega)$, critical perturbations either have the frequency ω_0 or are degenerate: they appear in pairs $\omega_0-\Delta\omega$, $\omega_0+\Delta\omega$ (see discussion by Crawford [1]). We restrict ourselves to nondegenerate case only.

For nearly critical behavior of small perturbations

$$\rho(\omega,\varphi,t)=\frac{1}{2\pi}\left[1+\frac{\pi\varepsilon_0A_1(t)}{D+i(\omega-\Omega)}e^{i(\varphi-\Omega t)}+cc\right.$$
$$\left.+\frac{\pi^2\varepsilon_0^2A_1^2(t)}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}e^{i2(\varphi-\Omega t)}+cc+O(A_1^3)\right],$$

the order parameter is $R(t)=2\pi A_1^*e^{i\Omega t}+O(A_1^3)$, and Eq. (10) holds with

$$\lambda_2(\varepsilon,\Omega)=\frac{\varepsilon_2}{i\pi\varepsilon^2G'(\Omega+iD)}, \qquad (13)$$

$$P(\varepsilon,\Omega)=\frac{\pi^2\varepsilon^2}{D}\left[1+\frac{G(\Omega+iD)-G(\Omega+2iD)}{iDG'(\Omega+iD)}\right]+\frac{i4\pi\varepsilon_{of}e^{i(2\Omega T-\nu)}}{\varepsilon G'(\Omega+iD)}\,. \qquad (14)$$

For **Lorentzian distribution of natural frequencies** the characteristic equation $\Lambda(\lambda)=0$ has *only one root*; and the bifurcation of the non-synchronous state is a *Hopf* one at $\varepsilon_0=2(\gamma+D)$ with the frequency $\Omega=\omega_0$. So,

$$\lambda_2(\varepsilon_0,\omega_0)=\frac{\varepsilon_2}{2}, \quad P(\varepsilon_0,\omega_0)=\frac{1}{2D+\gamma}-4\pi^2\varepsilon_{of}e^{-i\nu}(\gamma+D).$$

The real part of P determines, according to (10), the amplitude of the establishing collective mode $|A_1|^2=\lambda_2(\Re P)^{-1}$, with

$$\Re P(\varepsilon_0,\omega_0)=\frac{1}{2D+\gamma}-4\pi^2\varepsilon_{of}(\gamma+D)\cos(\nu).$$

One can see that depending on the value of ν, the amplitude decreases or increases due to additional nonlinear feedback. Moreover, for strong enough feedback $\Re P$ can become *negative*, what means a *subcritical* Kuramoto transition. Also, a nonlinear shift of the rotation frequency of R in the counterclockwise direction appears

$$\omega_2=\frac{\varepsilon_2\Im(P)}{2\Re(P)}=\frac{\varepsilon_2}{2}\frac{\tan\nu}{\left[4\pi^2\varepsilon_{of}(2D-\gamma)(D+\gamma)\cos\nu\right]^{-1}-1}.$$

4. Conclusions

We have developed a weakly nonlinear analysis of the effect of delayed feedback on the Kuramoto transition. In particular,

— We show that a linear delayed feedback not only controls the transition point, but effectively changes the nonlinear terms near the transition;

— A purely nonlinear delayed coupling does not effect the transition point, but can reduce or enhance the amplitude of collective oscillations.

We have restricted our attention to the most general case of Hopf bifurcation and have not considered other types of transition that occur under certain symmetries. The analysis is, of course, restricted to a vicinity of the transition point, moreover, the basic phase-coupling model assumes that all type of coupling are weak. A strong coupling case should be studied numerically.

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References

[1] J.D.Crawford, J. Stat. Phys. **74**(5/6), 1047 (1994).

[2] P.A.Tass. *Phase Resetting in Medicine and Biology. Stochastic Modelling and Data Analysis*. Springer, Berlin, 1999. D.Golomb, D.Hansel, and G.Mato. In F.Moss and S.Gielen, editors, *Neuro-informatics and Neural Modeling*, volume 4 of *Handbook of Biological Physics*, pages 887–968. Elsevier, Amsterdam, 2001. J.Milton and P.Jung, editors. *Epilepsy as a Dynamic Disease*. Springer, Berlin, 2003.

[3] M.G.Rosenblum and A.S.Pikovsky, PRE **70**, 041904 (2004); PRL **92**, 114102 (2004).

[4] O.V.Popovych, Ch.Hauptmann, and P.A.Tass, PRL **94**, 164102 (2005); Biol.Cybern. **95**, 69 (2006).